GRAPHICAL MODELS AND IMAGE PROCESSING
Vol. 59, No. 5, September, pp. 302–309, 1997
ARTICLE NO. IP970427

Discrete Analytical Hyperplanes

Eric Andres,* Raj Acharya,†,[1] and Claudio Sibata*

*Rosewell Park Cancer Institute, Radiation Medicine Department, Elm & Carlton Streets, Buffalo, New York 14263; and
†ECE Department, SUNY Buffalo, Buffalo, New York 14260

Received October 20, 1995; revised December 20, 1996; accepted March 18, 1997

This paper presents the properties of the discrete analytical hyperplanes. They are defined analytically in the discrete domain by Diophantine equations. We show that the discrete hyperplane is a generalization of the classical digital hyperplanes. We present original properties such as exact point localization and space tiling. The main result is the links made between the arithmetical thickness of a hyperplane and its topology. © 1997 Academic Press

1. INTRODUCTION

In the past 30 years, since the first 2D line digitization algorithm published by Bresenham [6], many papers have dealt with digital geometry but usually only in dimensions 2 and 3. Comparatively few works have handled higher dimensions. Much attention has been put into the digitization of 2D [2, 7] or 3D primitives [2, 12, 15–17]. A *primitive* is an elementary geometrical object such as a line, a circle, a polygon, a spline, or a Bezier patch. It is important to be able to handle correctly primitives because, typically, a real world object is represented inside a computer as a union, intersection, or difference of analytically defined continuous primitives. This is what we call the *continuous analytical representation* (CAR) form. For instance, the CAR of a hyperplane in dimension n would be "$\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i = 0$." This representation form is interesting because the manipulation of objects represented in a CAR form is easy. Another classical way of representing real world objects is the *discrete representation* (DR) form. An object is simply represented as a set of discrete points. The problem of this representation form is that it is very difficult to handle. The Euclidean and the discrete worlds do not follow the same geometrical and topological laws: for instance, two orthogonal discrete 2D lines could have no intersection points. However, it is an important type of object representation that is used in many different fields such as visualization [8], medical imaging [19], computer-guided surgery [1], discrete ray tracing [23], and some flight simulators [9]. There are two main ways of obtaining a DR of a real world object. One is by acquisition with a physical device like a camera for a 2D DR, a CT or MR scanner for a 3D DR, a PET scanner for a 4D DR, etc. The second one is by digitizing the primitives forming its CAR.

Although many papers have dealt with the digitization of primitives, the problems we have in handling the geometry and topology of the discrete world resulted in a predominant approach: "a digital primitive is the result of a local approximation process applied to the continuous primitive." The local approximation process is called a *digitization scheme*. The advantages of the digitization scheme approach are clear: it is a simple method that has been successfully applied to many different types of primitives [15, 16]. However, there are also some major drawbacks. Digitization schemes are difficult to generalize to higher dimensions. A digital primitive is defined through a local approximation process. This makes it difficult to determine and control the properties, especially global properties, of the digital primitive.

In this paper another approach is proposed. A hybrid representation form called *discrete analytical representation* (DAR) form is used to study a new discrete hyperplane introduced by Reveillès [20]. The new hyperplane is called the *discrete analytical hyperplane*. The discrete analytical hyperplane is not defined as an approximation of a continuous hyperplane through a digitization scheme but is defined directly in the discrete world by an analytical definition, a double Diophantine inequation. For instance, the CAR of a hyperplane in dimension n is "$0 \le \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i < \omega$," where ω is the *arithmetical thickness* of the discrete hyperplane. The major advantage of defining a discrete primitive in a DAR form is that the definition is global. The analytical definition allows an easy study of the primitive and a better control of its properties.

From the analytical definition of the discrete analytical hyperplane some fundamental properties can be deduced. Determining if a point belongs or not or if it is on one side or another of a discrete analytical hyperplane is immediate. A discrete analytical hyperplane tiles the space and

[1] To whom correspondence should be addressed.

1077-3169/97 $25.00

a discrete hyperplane is tiled by discrete hyperplanes of inferior dimensions. Some other minor results are also immediately deduced from the definition. An important part of this paper concerns the control of the topology of a discrete hyperplane by its arithmetical thickness. A necessary and sufficient condition is proposed for which a discrete analytical hyperplane is k-separating (has no k-tunnels) and minimal (no simple points). Some limited results on the connectivity of a hyperplane are deduced from this results. We finish the paper by showing that the discrete analytical hyperplane is a generalization of the Bresenhams discrete 2D line of Bresenham and of Stojmenovics and Veerlaerts digital hyperplanes obtained by digitization schemes [21, 22].

In Section 2, after a short preliminary presention of the notations used in this paper, the analytical definition of the discrete analytical hyperplane is introduced followed by the first properties of the hyperplane. Section 3 is devoted to the arithmetical thickness and the control it allows on the topology of the hyperplane. In Section 4 we show that the digital hyperplanes introduced by Stojmenovic [21] and Veerlert [22] are particular discrete analytical hyperplanes. We conclude in Section 5 with some comments and open questions.

2. DEFINITIONS AND FIRST PROPERTIES

2.1. *Preliminaries*

The set of the real numbers is noted $\mathbb{R}$, the set of the integers $\mathbb{Z}$, and the set of the strictly positive integers $\mathbb{N}^*$. $\lfloor x \rfloor$ is the greatest integer smaller or equal to x, with x a rational or real value (instruction "floor" in the C computer language) [3]. If not specified differently $\{x\}$, where x is a real or rational value, is the *Euclidean remainder* $\{x\} = x - \lfloor x \rfloor$ and not a single element set. $\gcd(\alpha_1, \ldots, \alpha_n) = u$ is the greatest common divisor of the integers $\alpha_1, \ldots, \alpha_n$. $S \uplus T$ stands for $S \cup T$, where $S \cap T = \emptyset$. In this article n represents the dimension of the space.

A *discrete point* is a point in $\mathbb{Z}^n$. A *digital point* is a point with integer coordinates in $\mathbb{R}^n$. A *discrete* (*digital*) *object* is a set of discrete (digital) points. For S a discrete object, $\overline{S} = \mathbb{Z}^n \backslash S$. Two discrete points $A(a_1, \ldots, a_n)$ and $B(b_1, \ldots, b_n)$ are *k-neighbors* if $|a_i - b_i| \le 1$ for $i \in [1, n]$ and $k \le n - \sum_{i=1}^{n} |a_i - b_i|$. This definition corresponds to the following classical notation: 2D two 0-neighbors (respectively, 1-neighbors) are classically called 8-connected (respectively, 4-connected) neighbors. In 3D two 0-neighbors (respectively, 1-neighbors, 2-neighbors) are classically called 26-connected (respectively, 18-connected, 6-connected) neighbors (see Fig. 2). For a discrete object S, A *k-path in S* is a sequence of discrete points all in S such that consecutive pairs of points are k-neighbors. A discrete object S is *k-connected* if there is a k-path in S between two arbitrary points in S. A *k-connected component* is a maximal k-connected set. For T a subset of a discrete object S, T is *k-separating* in S if $S \backslash T$ is not k-connected. A discrete object is said to be k-separating if it is k-separating in $\mathbb{Z}^n$. Let S be a k-separating discrete object such that $\overline{S}$ has exactly two k-connected components. A *k-simple point p* in S is a discrete point such that $S \backslash \{p\}$ is k-separating. A *simple point* in S is a k-simple point in S for some k. A k-separating object is called *k-minimal* if it does not contain any k-simple points.

A *voxel* $V(A)$ in $\mathbb{R}^n$ corresponding to a discrete point $A(a_1, \ldots, a_n)$ is defined by the continuous hypercube $V(A) = [a_1 - \frac{1}{2}, a_1 + \frac{1}{2}] \times \cdots \times [a_n - \frac{1}{2}, a_n + \frac{1}{2}]$.

2.2. *Analytical Definition*

The definition of a discrete analytical hyperplane $P = P_n(\alpha_0, \ldots, \alpha_n, \omega)$ in dimension n is now introduced (see Fig. 1 for illustration in 2D). ω is called the arithmetical thickness, $\alpha_1, \ldots, \alpha_n$ are called the *coefficients* and α_0 is called the *translation constant* of the hyperplane. The control value of P in a discrete point X $(x_1, \ldots, x_n)$ is defined by $\Pi(P, X) = \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i$. We have:

Definition 1 (Discrete analytical hyperplane). A discrete analytical hyperplane $P = P_n(\alpha_0, \ldots, \alpha_n, \omega)$ in dimension n is defined by the double Diophantine inequality:

$$X \in P \text{ if and only if } 0 \le \Pi(P, X) < \omega,$$

where $n \in \mathbb{N}^*$, $X \in \mathbb{Z}^n$, $(\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^{n+1}$, $\omega \in \mathbb{N}^*$ and $\gcd(\alpha_1, \ldots, \alpha_n) = 1$.

The discrete hyperplane has been introduced by Reveillès [20]. Two hyperplanes are said to be *equivalent* [20] if they differ only by their translation constant. A *hyperplane segment* is a connected subset of a hyperplane. A hyperplane is called a *naive hyperplane* if $\omega = \max_{i=1..n}(|\alpha_i|)$ and a *standard hyperplane* if $\omega = \sum_{i=1}^{n} |\alpha_i|$. The term "naive" was introduced by Reveillès in [20] and a study of the naive 3D plane can be found in [11] (see also Fig. 4). The term "standard" was introduced by Françon in [13] (see also Fig. 8).

We assume without loss of generality that $\gcd(\alpha_1, \ldots, \alpha_n) = 1$, because if $\gcd(\alpha_1, \ldots, \alpha_n) = a \ge 1$, then $P_n(\alpha_0, \ldots, \alpha_n, \omega) = P_n(\lfloor \alpha_0/a \rfloor, \alpha_1/a, \ldots, \alpha_n/a, \omega')$, where $\omega' = \lfloor \omega/a \rfloor + 1$ if $\{\omega/a\} > \{\alpha_0/a\}$ and $\omega' = \lfloor \omega/a \rfloor$ else.

An important part of the present study concerns the tunnels of the hyperplanes:

Definition 2 (k-tunnel of a hyperplane). A hyperplane $P = P_n(\alpha_0, \ldots, \alpha_n, \omega)$ has a k-tunnel $T(X, Y)$ if there are two k-neighbors A and B, such that: $\Pi(P, A) < 0$ and $\Pi(P, B) \ge \omega$.

If a hyperplane has k-tunnels, it has also $(k-1)$-tunnels.

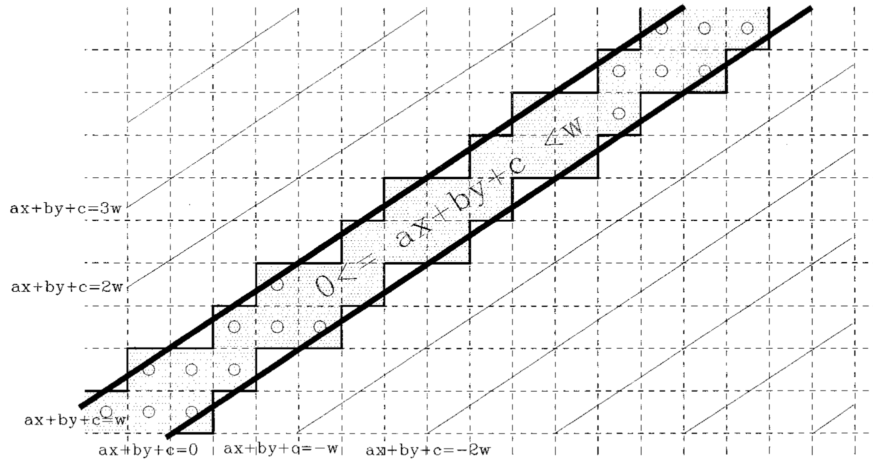

FIG. 1. Diophantine definition of a 2D line and space tiling.

A hyperplane that has no k-tunnels is k-separating. A 0-separating hyperplane is said to be *tunnel free.*

2.3. *First Properties of the Hyperplanes*

Some simple but important properties can be immediately deduced from the definition of a discrete hyperplane. Most of these properties are properties introduced by Reveillès for the 3D plane and are here generalized to arbitrary dimensions.

PROPOSITION 3 (Exact point localization). *For a hyperplane* $P = P_n(\alpha_0, \ldots, \alpha_n, \omega)$, *a discrete point* $X(x_1, \ldots, x_n)$:

- *belongs to the hyperplane if and only if* $0 \leq \Pi(P, X) < \omega$,
- *is on one side of the hyperplane if and only if* $\Pi(P, X) < 0$,
- *and on the other side if and only if* $\Pi(P, X) \geq \omega$.

This result is obvious with the Diophantine definition, but not with the classical approach. Usually the digitization scheme has no inverse, so determining if a point belongs to a digital object is often complicated.

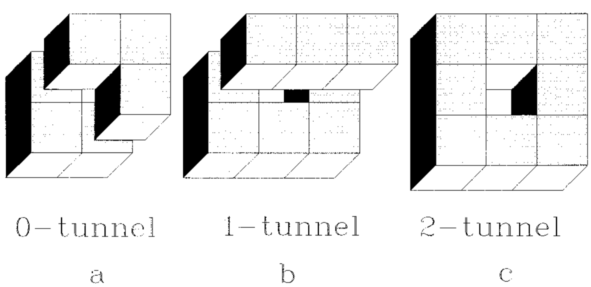

FIGURE 2

PROPOSITION 4 (Inclusion). $P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega) \subset P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega')$ *if* $\omega < \omega'$; *and more precisely, if* $\omega' = \omega + \omega_0$ *then:*

$$P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega') = P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega) \uplus P_n(\alpha_0 - \omega, \alpha_1, \ldots, \alpha_n, \omega_0).$$

PROPOSITION 5 (Tiling of $\mathbb{Z}^n$). $\mathbb{Z}^n \equiv \biguplus_{t=-\infty}^{+\infty} P_n(\alpha_0 + t\omega, \alpha_1, \ldots, \alpha_n, \omega)$.

PROPOSITION 6 (Recurrent form of hyperplane in dimension n). $P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega) = \biguplus_{x_n=-\infty}^{+\infty} \{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{Z}^n \mid (x_1, \ldots, x_{n-1}) \in P_{n-1}(\alpha_0 + \alpha_n x_n, \alpha_1, \ldots, \alpha_{n-1}, \omega)\}$.

Proposition 6 tells us that a slice orthogonal to an axis of a hyperplane, of dimension n, is a hyperplane of dimension $n - 1$.

COROLLARY 7 (Developed recurrent form of a hyperplaneplane). $P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega) = \biguplus_{x_3=-\infty}^{+\infty} \cdots \biguplus_{x_n=-\infty}^{+\infty} \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid (x_1, x_2) \in P_2(\alpha_0 + \sum_{i=3}^{n} \alpha_i x_i, \alpha_1, \alpha_2, \omega)\}$.

Corollary 7 shows the developed recurrent form of a hyperplane in dimension n. All the properties of a hyperplane can be related to the properties of 2D lines. Note that in Proposition 6 and its corollary, the choice of an axis x_i is arbitrary. These properties are independent of the choice of axis. There is no major axis considered contrary to [21] (see Section 4).

PROPOSITION 8 (Periodicity). *If* $X(x_1, \ldots, x_n)$ *is a point of* $P = P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega)$ *then, for* $1 \leq i, j \leq n$ *and* $k \in \mathbb{Z}$; $X'(x_1, \ldots, x_i + k\alpha_j, \ldots, x_j - k\alpha_i, \ldots, x_n)$ *is also a point of P with same control value as in X.*

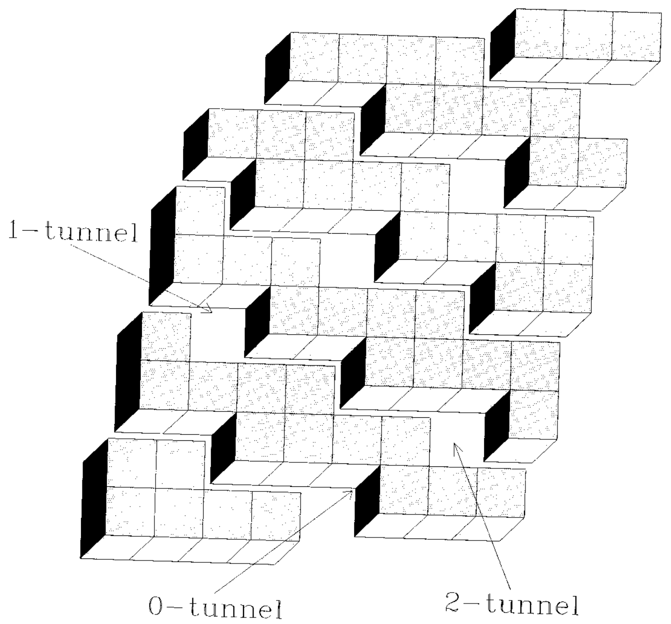

FIG. 3. Discrete 3D plane $2x + 5y - 9z$ with arithmetical thickness $\omega = 8$.

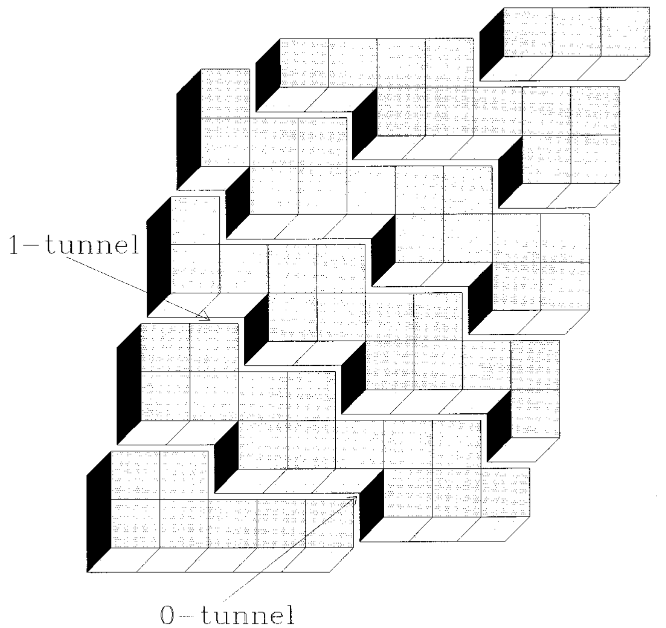

FIG. 4. Discrete 3D plane $2x + 5y - 9z$ with arithmetical thickness $\omega = 9$.

Propositions 3–6, and 8 are obvious. In Section 5, we will consider the hyperplanes proposed through the classical digitization approach and show that the classical digital hyperplanes are particular cases of the discrete analytical hyperplanes. Despite the fact that these properties are therefore also verified for the digital hyperplanes, their definition makes it difficult to formulate and prove such properties.

3. ARITHMETICAL THICKNESS

We will now discuss some of the advantages of the arithmetical thickness in the definition of the discrete hyperplane. The arithmetical thickness is a new concept that was introduced by Reveillès. There is no equivalent of the arithmetical thickness in the classical digital hyperplane definitions based on digitization schemes as we will see in Section 4.

One of the main results of this paper is the link made between the arithmetical thickness and the tunnels of the hyperplane:

Proposition 9 (Arithmetical thickness and tunnels). *Let $P = P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega)$ be a discrete hyperplane where $0 \le \alpha_i \le \alpha_{i+1}$ for all i.*

- *if $\omega < \alpha_n$, the hyerplane has $(n-1)$-tunnels*:
- *for $0 < k < n$, if $\sum_{i=k+1}^{n} \alpha_i \le \omega < \sum_{i=k}^{n} \alpha_i$ the hyperplane has $(k-1)$-tunnels and is k-separating*;
- *if $\omega \ge \sum_{i=1}^{n} \alpha_i$ the hyperplane is tunnel free.*

Figures 3 to 8 illustrate proposition 9 in 3D. The discrete analytical hyperplane $P = P_n(0, 2, 5, -9, \omega)$ has 2-tunnels for $\omega = 8 < |-9|$ (Fig. 3). There are no more 2-tunnels for $\omega = 9$, but there are still 1-tunnels (Fig. 4). For $\omega = 13 = |-9| + |5| - 1$, P still has 1-tunnels (Fig. 5), but not anymore when $\omega = 14 = |-9| + |5|$ (Fig. 6). The plane P has

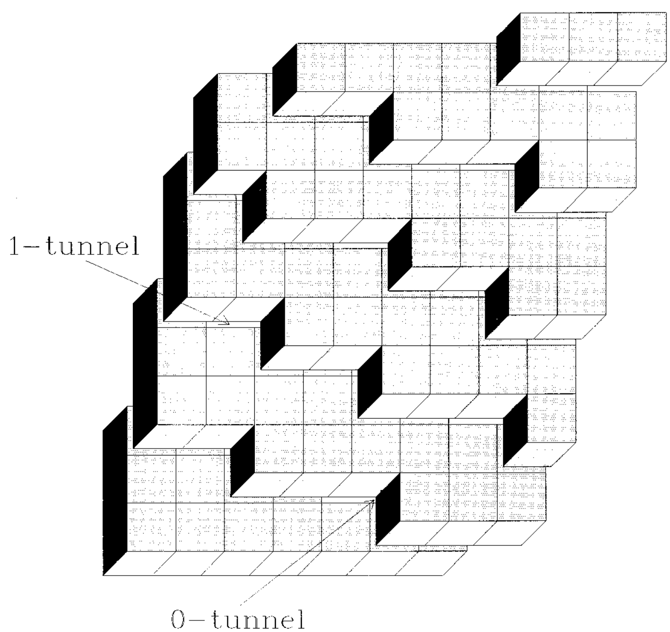

FIG. 5. Discrete 3D plane $2x + 5y - 9z$ with arithmetical thickness $\omega = 13$.

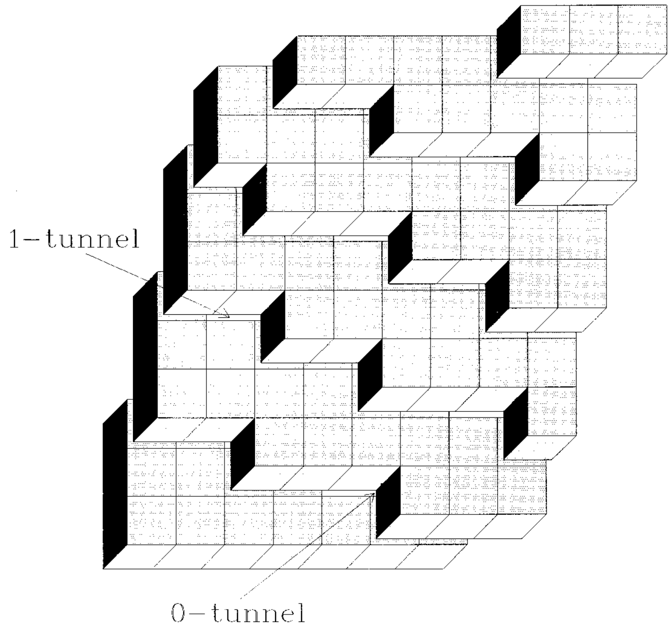

FIG. 6. Discrete 3D plane $2x + 5y - 9z$ with arithmetical thickness $\omega = 14$.

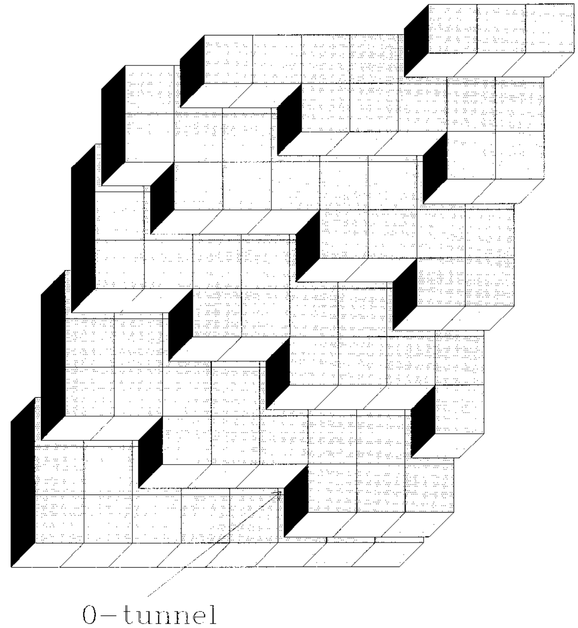

FIG. 8. Discrete 3D plane $2x + 5y - 9z$ with arithmetical thickness $\omega = 16$.

0-tunnels for $\omega = 14$ and for $\omega = 15 = |-9| + |5| + |2| - 1$ (Fig. 7) and is tunnel-free for $\omega = 16 = |-9| + |5| + |2|$ (Fig. 8).

Proof of Proposition 9. Let's note first that the condition "$0 \le \alpha_i \le \alpha_{i+1}$" can be taken without loss of generality because with simple symmetries we can always come back

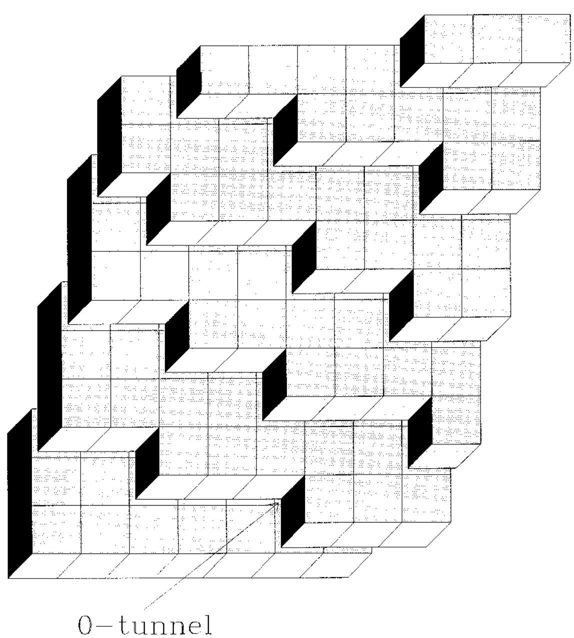

FIG. 7. Discrete 3D plane $2x + 5y - 9z$ with arithmetical thickness $\omega = 15$.

to these conditions. We want to show that $\omega = \sum_{i=k+1}^{n} \alpha_i$ is the smallest value for which there are no more k-tunnels. For that, let us first show that there is a least one k-tunnel for $\omega = \sum_{i=k+1}^{n} \alpha_i - 1$. Since $\gcd(\alpha_1, \ldots, \alpha_n) = 1$, there exists $Y(y_1, \ldots, y_n)$ such that $\sum_{i=1}^{n} \alpha_i y_i = 1$ [14]. Let us consider the discrete point $A(a_1, \ldots, a_n)$ with $a_i = -y_i$. We have $\Pi(P, A) = -1$. Let's consider the discrete point $B(b_1, \ldots, b_n)$ with $b_i = a_i$ for $1 \le i \le k$ and $b_i = a_i + 1$ for $k + 1 \le i \le n$. By construction A and B are k-neighbors. We have $\Pi(P, B) = \sum_{i=1}^{n} \alpha_i b_i = \sum_{i=1}^{n} \alpha_i a_i + \sum_{i=k+1}^{n} \alpha_i = \omega$. This proves that there is a k-tunnel in a hyperplane for the arithmetical thickness $\omega = \sum_{i=k+1}^{n} \alpha_i - 1$.

Let us show now that there is no more k-tunnel for $\omega = \sum_{i=k+1}^{n} \alpha_i$. Let us consider two discrete points $A(a_1, \ldots, a_n)$ and $B(b_1, \ldots, b_n)$ such that $\Pi(P, A) = -1$ and B is a k-neighbor of A. B being a k-neighbor of A means that $b_i = a_i + e_i$, where $|e_i| \le 1$ and $\sum_{i=1}^{n} |e_i| \le n - k$. So $\Pi(P, B) = \sum_{i=1}^{n} \alpha_i a_i + \sum_{i=1}^{n} \alpha_i e_i \le -1 + \sum_{i=k+1}^{n} \alpha_i$ and finally $\Pi(P, B) \le \omega - 1$. This shows that B cannot be on the other side (than A) of P and therefore that there is no k-tunnel for $\omega = \sum_{i=k+1}^{n} \alpha_i$. ■

Proposition 9 shows that contrary to what happens for the discrete analytically defined hyperspheres [5], the arithmetical thickness provides an optimal control over the tunnels of a hyperplane. The condition "$\omega \ge \sum_{i=k+1}^{n} \alpha_i$" is necessary and sufficient for a hyperplane to be k-separating. In the case of the hyperspheres the arithmetical thickness provides conditions on the k-separation that are sufficient but not necessary [5]. Proposition 9 is important because it makes the link between the geometry and topology of a hyperplane and its arithmetical thickness. It becomes therefore easy to choose the thickness of the hyperplane needed for a given application [4, 10, 11, 13, 18].

COROLLARY 10 (Limited results on the connectivity). *Let $P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega)$ be a discrete hyperplane of dimension $n \geq 3$ where $0 \leq \alpha_i \leq \alpha_{i+1}$ for all i.*

- *for $2 \leq k < n$, if $\sum_{i=k+1}^{n} \alpha_i < \omega < \sum_{i=k}^{n} \alpha_i$ the hyperplane is $(n-k)$-connected or $(n-k+1)$-connected;*
- *for $2 \leq k < n$, if $\omega = \sum_{i=k+1}^{n} \alpha_i$ the hyperplane is $(n-k)$-connected and not $(n-k+1)$-connected;*
- *for $\omega \geq \sum_{i=2}^{n} \alpha_i$ the hyperplane is $(n-1)$-connected.*

Corollary 10 is deduced from Proposition 9. The real connectivity of a hyperplane is not always known. Only a lower bound of the connectivity is given for the hyperspheres of arithmetical thickness $\sum_{i=k+1}^{n} \alpha_i \leq \omega < \sum_{i=k}^{n} \alpha_i$. The real connectivity is a more complicated arithmetical problem that has no complete answer for the moment, to the best of the author's knowledge, except in 2D [20] as presented by Corollary 11.

COROLLARY 11 (Connectivity of a 2D line [20]). *Let $P_2(\alpha_0, \alpha_1, \alpha_2, \omega)$ be a 2D line where $0 \leq \alpha_1 \leq \alpha_2$.*

- *if $\omega < \alpha_2$ the line is not connected;*
- *if $\alpha_2 \leq \omega < \alpha_1 + \alpha_2$ the line is 0-connected;*
- *if $\omega \geq \alpha_1 + \alpha_2$ the line is 1-connected.*

PROPOSITION 12 (k-minimality). *Let $P = P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega)$ be a discrete hyperplane where $0 \leq \alpha_i \leq \alpha_{i+1}$ for all i, and $\omega = \sum_{i=k+1}^{n} \alpha_i$. P is k-minimal.*

Figure 4 shows a 3D plane that is 2-minimal. Figure 6 shows a 3D plane that is 1-minimal, and Fig. 8 shows a plane that is 0-minimal.

Proof of Proposition 12. Let us consider a hyperplane $P = P_n(\alpha_0, \alpha_1, \ldots, \alpha_n, \omega)$, with $0 \leq \alpha_i \leq \alpha_{i+1}$, $\omega = \sum_{i=k+1}^{n} \alpha_i$ and an arbitrary point $A(a_1, \ldots, a_n)$ of P. Let us prove that A is not a simple point of P. Proposition 9 tells us that P is k-separating. Let us show that $P \backslash \{A\}$ is not k-separating. Let us consider the discrete points $B(b_1, \ldots, b_n)$ and $C(c_1, \ldots, c_n)$, where $b_i = c_i = a_i$ for all $1 \leq i \leq k$, and $b_i = a_i - 1$ and $c_i = a_i + 1$ for all $i > k$.

We have $\Pi(P, B) = \alpha_0 + \sum_{i=1}^{n} \alpha_i a_i - \sum_{i=k+1}^{n} \alpha_i = \Pi(P, A) - \omega < 0$. In the same way we have $\Pi(P, C) \geq \omega$. This shows that B and C are on opposite sides of P. By construction both are k-neighbors of A, this means that if we remove A from P we create a k-path from one side to the other and therefore $P \backslash \{A\}$ is not k-separating. ■

4. DIGITAL HYPERPLANES AND DISCRETE HYPERPLANES

In [21], Stojmenovic and Tosic proposed a digitization scheme for Euclidean hyperplanes. This digitization scheme has been improved by Veerlaert in [22]. We will show that the digital hyperplanes resulting from their digitization scheme are naive hyperplanes if we deal with rational instead of irrational numbers. This result is a generalization of a previous result [11] to arbitrary dimensions.

Let us first present the digitization scheme and digital hyperplane of Stojmenovic [21]. We present it with the notations used in our paper.

DEFINITION 13 (Stojmenovics' digitization scheme). Let us consider an Euclidean hyperplane P in $\mathbb{R}^n$ defined by $\gamma_0 + \sum_{i=1}^{n} \gamma_i x_i = 0$, with $|\gamma_i| \leq |\gamma_n|$ for $1 \leq i \leq n-1$, and $\gamma_n > 0$. The axis of x_n is called the major axis of P. Let the crossing point of P and a coordinate digital straight line parallel to the major axis x_n of P be the point $p = (r_1, \ldots, r_{n-1}, p_n)$, with r_i integer and p_n real.

Then the point $p' = (r_1, \ldots, r_{n-1}, r_n)$ where $p_n - \frac{1}{2} < r_n \leq p_n + \frac{1}{2}$, is the digital image of p.

The set consisting of all points p' digital images of points of P is the digital image of P, denoted P'.

Definition 14 presents the notion of digital flatness proposed by Veerlaert in [22].

DEFINITION 14 (Digital flatness). Let $S \subset \mathbb{Z}^n$. S is flat if and only if there exists $n + 1$ real $\gamma_0, \ldots, \gamma_n$ such that:

- $\max\{|\gamma_i|;\ 1 \leq i \leq n\} = 1$
- every point $A = (a_1, \ldots, a_n)$ of S satisfies $-\frac{1}{2} < \gamma_0 + \sum_{i=1}^{n} \gamma_i a_i \leq \frac{1}{2}$.

Veerlaert uses the digitization scheme defined by Stojmenovic (definition 13 and [21]) and shows that the so-defined digital hyperplane is a digital flatness and conversely that a digital flatness is a segment of a digital hyperplane. It is interesting to see that Veerlaert does not introduce a new digitization scheme but uses the result of Stojmenovic's digitization scheme and proposes a mathematical formulation for the points of the digital hyperplane.

PROPOSITION 15 (Digital hyperplanes are discrete naive hyperplanes). *Stojmenovic's digital hyperplane is a naive discrete analytical hyperplane if the coefficients of the Euclidean digitized hyperplane are rational numbers.*

The restriction to rational numbers for the definition of the discrete hyperplane, and for Stojmenovic's digital hyperplanes, is not a real restriction since we are usually interested in finite hyperplane segments.

Proof of Proposition 15. Veerlaert showed in [22] that a digital hyperplane is a digital flatness and that a digital flatness is a digital segment of a digital hyperplane. Let us first show that a discrete hyperplane $P = P(\alpha_0, \ldots, \alpha_n, \omega)$, with $\omega = \max\{|\alpha_i|; 1 \leq i \leq n\}$ is flat. P is a naive hyperplane.

We have $P = P(\alpha_0, \ldots, \alpha_n, \omega) = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid -(\omega/2) < -\alpha_0 + (\omega/2) - \sum_{i=1}^{n} \alpha_i x_i \leq (\omega/2)\}$ which is obviously flat since $\omega = \max\{|\alpha_i|;\ 1 \leq i \leq n\}$. This shows that a discrete naive hyperplane is a digital hyperplane as defined by Stojmenovic and Veerlaert.

Let us now show that a digital hyperplane is a discrete naive hyperplane if the digital hyperplane corresponds to a Euclidean hyperplane defined with rational numbers.

Let us consider a Euclidean hyperplane P defined by $\gamma_0 + \sum_{i=1}^{n} \gamma_i x_i = 0$, with $\gamma_i = (\alpha_i/|\alpha_n|)$, $\alpha_i \in \mathbb{Z}$, where $|\alpha_n| = \max\{|a_i|; 1 \leq i \leq n\}$.

Let us consider Stojmenovic's cooresponding digital hyperplane P'. Veerlaert shows in [22] that this digital hyperplane verifies $-\frac{1}{2} < \gamma_0 + \sum_{i=1}^{n} \gamma_i a_i \leq \frac{1}{2}$, or $-\frac{1}{2} \leq \gamma_0 + \sum_{i=1}^{n} \gamma_i a_i < \frac{1}{2}$, for every point $A = (a_1, \ldots, a_n)$ of P'.

We have the discrete analytical hyperplane

$$P'' = P_n\left(-\alpha_0 + \frac{|\alpha_n|}{2}, -\alpha_1, \ldots, -\alpha_n, |\alpha_n|\right)$$

$$= \left\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \,|\, 0 \leq -\alpha_0 + \frac{|\alpha_n|}{2} + \sum_{i=1}^{n} (-\alpha_i x_i) < |\alpha_n|\right\}$$

$$= \left\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \,|\, -\frac{|\alpha_n|}{2} \leq -\alpha_0 + \sum_{i=1}^{n} (-\alpha_i x_i) < \frac{|\alpha_n|}{2}\right\} = P'. \quad \blacksquare$$

PROPOSITION 16 (Generalization of Bresenham's line). *The discrete analytical hyperplane is a generalization of Bresenham's digital line to an arbitrary dimension and arbitrary arithmetical thickness.*

Proof of Proposition 16. Reveillès showed in [20] that a discrete line $P_2(c, a, b, \omega)$ with $c = \lfloor b/2 \rfloor + 1$ and $\omega = b$ [20] is a Bresenham line [6] and the discrete hyperplane is an immediate generalization of the discrete line to arbitrary dimensions. It is also clear that Stojmenovic's digitization scheme is a generalization of Bresenham's digitization scheme but only to arbitrary dimensions and not to arbitrary thicknesses. ∎

5. CONCLUSION

In this paper we have applied the "arithmetical geometry" approach proposed by Reveillès [20] to the study of the discrete hyperplanes. We provide a simple mathematical definition and describe different properties like exact point localization, space, and hyperplane tiling. We describe the recurrent construction form for the discrete hyperplanes that should allow the design of an efficient hyperplane generation algorithm based on the generation of a single 2D discrete line. The link between topology of the discrete hyperplane, tunnels in the hyperplane and arithmetical thickness is made. We finish by showing that the discrete hyperplane is a generalization of the digital hyperplanes obtained by the classical "digitization scheme" approach.

The discrete analytical representation form of primitives has led to interesting new properties for the discrete hyperspheres [4] and, here, for the discrete hyperplanes. It allows also an easy extension of basic 2D primitives to higher dimensions. This is obtained by avoiding the problematic "digitization scheme" step. Nevertheless, some difficulties exist when we try to define discrete m-flats in n-dimensional space ($m \leq n - 1$) [21]. The simplest example is the 3D line. These objects cannot be defined just by intersection of hyperplanes: the intersection of two 3D planes usually cannot be considered a 3D line, like the intersection of two discrete lines in 2D is usually not a point [11, 20]. The first thought that comes to mind is to extend the notion of arithmetical thickness and to define a cylinder around a continuous 3D line in order to define a discrete 3D line, but that approach has not provided us with interesting results. The space is anisotropic and there does not seem to be a satisfying relationship between the radius of the cylinder and the topology of the discrete 3D line. Some other solutions need to be studied. Another such example are discrete ellipses which also cannot be described so simply by Diophantine inequalities. This will require further research in discrete geometry. Much work remains to be done on the discrete hyperplanes; e.g., designing an efficient generation algorithm, defining a n-dimensional polyhedra, hyperplane recognition, and hyperplane intersection computation.

ACKNOWLEDGMENTS

We particularly thank Dr. D. Cohen, Professor J. Françon, and the reviewer.

REFERENCES

1. L. Adams *et al.*, Computer assisted surgery, *IEEE Comput. Graphics* **10**(3), May 1990, 43–51.
2. E. Andres, Discrete circles, hyperspheres and spheres, *Comput. Graphics* **18**(5), 1994, 695–706.
3. E. Andres, Choice of integer part function for computer graphics, *IEEE TVCG*, submitted.
4. E. Andres and C. Sibata, New methods in Oblique slice generation, in *SPIE's Int. Symp. on Medical Imaging '96, Feb. 1996*, Vol. 2707, pp. 580–589.
5. E. Andres, C. Sibata, and R. Acharya, General analytical hyperspheres, *IEEE TVCG*, submitted.
6. J. E. Bresenham, Algorithm for computer control of a digital plotter, *IBM Syst. J.* **4**(1), 1965, 25–30.
7. J. E. Bresenham, A linear algorithm for incremental digital display of circular arcs, *Comm. ACM* **20**(2), 1977, 100–106.
8. J. L. Coatrieux and C. Barillot, A survey of 3D display techniques

to render medical data, *3D Imaging in Medicine* (K. H. Hohne, H. Fuchs, and S. K. Pizer, Eds.), pp. 175–195, Springer-Verlag, Berlin, 1990.

9. D. Cohen and A. Shaked, Photo-realistic imaging of digital terrains, *Comput. Graphics Form* **12**(3), 1993, 363–373.
10. I. Debled-Renesson and J-P. Reveilles, An incremental algorithm for digital plane recognition, in *4th Discrete Geometry for Computer Imagery Conference, Grenoble, France, Sept. 1994*.
11. I. Debled-Renesson and J-P. Reveilles, A new approach to digital planes, in *SPIE Vision Geometry III, Boston, Oct.–Nov. 1994*, Vol. 2356.
12. S. Forchhammer, Digital plane and grid point segments, *Comput. Vision Graphics Image Process* **47,** 1989, 373–389.
13. J. Francon, Discrete combinatorial surfaces, CGVIP *Graphic Models Image Process* **57,** 1995.
14. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1979.
15. A. Kaufman and E. Shimony, 3D scan-conversion algorithms for voxel based graphics, in *Proc. ACM Workshop on interactive 3D graphics, Chapel Hill, NC, Oct. 1986*, pp. 45–76.
16. A. Kaufman, Efficient algorithms for 3D scan conversion of parametric curves, surfaces, volumes, *Comput. Graphics* **21**(4), 1987, 171–179.
17. C. E. Kim, Three-dimensional digital planes, *IEEE Trans. Pattern Anal. Machine Intell.* **5,** 1983, 231–234.
18. P. Nehlig and C. Montani, A discrete template based plane casting algorithm for volume viewing, in *5th DGCI, Clermont–Ferrand France, Sept. 1995*.
19. A. Pommert *et al.*, 3D-Imaging in Medicine, *Encyclopedia of Computer Science and Technology*, Vol. 28, Suppl. 13, pp. 341–370, 1993.
20. J-P. Reveilles, *Geometrie discrete, calcul en nombres entiers et algorithmique*, state thesis [in French], Universite Louis Pasteur, Strasbourg, Dec. 1991.
21. I. Stojmenovic and R. Tosic, Digitization schemes and the recognition of digital straight lines, hyperplanes and flats in arbitrary dimensions, *Vision Geometry*, Contempory Math. Series, Vol. 119, pp. 197–212, Amer. Math. Soc. Providence, RI, 1991.
22. P. Veerlaert, On the flatness of digital hyperplanes, *J. Math. Imaging Vision* **3,** 1993, 205–221.
23. R. Yagel, D. Cohen, and A. Kaufman, Discrete ray tracing, *IEEE Comput. Graphics* **12**(5), 1992, 19–28.